

# The Characteristic Polynomial Database

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A **family of Bohemian matrices** is a set of matrices of dimension  $n$  where the free entries are from the discrete set of integers of bounded height  $P$  called the **population** of the Bohemian family.

A **family of Bohemian matrices** is a set of matrices of dimension  $n$  where the free entries are from the discrete set of integers of bounded height  $P$  called the **population** of the Bohemian family.

**Bohemian eigenvalues** are the set of eigenvalues of a Bohemian family.

## Example Family 1

The family of  $6 \times 6$  matrices with population  $\{-1, +1\}$ .

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

## Example Family 2

The family of  $6 \times 6$  upper Hessenberg<sup>1</sup> matrices with population  $\{-1, +1\}$  and subdiagonal entries fixed at 1.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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<sup>1</sup>Chan et al., "Bohemian Upper Hessenberg and Toeplitz Matrices".

# Questions

- How many matrices are singular?
- What is the maximum determinant?
- How many distinct characteristic polynomials does the family have?
- How many distinct eigenvalues does the family contain?
- How many distinct Jordan canonical are there?

# How Many Matrices are Singular?

For the Bohemian family of  $6 \times 6$  matrices with population  $\{-1, +1\}$ ,  
how many matrices are singular?

## How Many Matrices are Singular?

By brute-force computation on  $2^{36} = 68,719,476,736$  matrices, there are **43,090,149,376** singular matrices.

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Language	Time
Maple	270 days

## How Many Matrices are Singular?

By brute-force computation on  $2^{36} = 68,719,476,736$  matrices, there are **43,090,149,376** singular matrices.

Language	Time
Maple	270 days
Matlab	10 days

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Language	Time
Maple	270 days
Matlab	10 days
Julia	31 hours

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Matlab	10 days
Julia	31 hours
Python (sequential)	20 days

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Matlab	10 days
Julia	31 hours
Python (sequential)	20 days
Python (batched)	17 hours

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Language	Time
Maple	270 days
Matlab	10 days
Julia	31 hours
Python (sequential)	20 days
Python (batched)	17 hours
C++	4.75 hours

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Language	Time
Maple	270 days
Matlab	10 days
Julia	31 hours
Python (sequential)	20 days
Python (batched)	17 hours
C++	4.75 hours
<b>CPDB</b>	<b>124ms</b>

# Symmetries

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Symmetry	Description	Reduction
Transpose	If $A$ is singular, $A^T$ must also be singular.	2
Negation	If $A$ is singular, $-A$ must also be singular.	2
Permutation	If $A$ is singular, $PAP^{-1}$ must also be singular for all permutation matrices $P$	$n!$

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# How Many Matrices are Singular?

For the Bohemian family of  $7 \times 7$  matrices with population  $\{-1, +1\}$ , how many matrices are singular?

## How Many Matrices are Singular?

By brute-force computation on  $2^{49} = 562,949,953,421,312$  matrices, there are ??? singular matrices.

Language	Time
Maple	7 millennia
Matlab	215 years
Julia	34 years
Python (sequential)	483 years
Python (batched)	21 years
C++	45 years



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## Matching 3D models with shape distributions

R Osada, T Funkhouser, B Chazelle... - ... Conference on Shape ..., 2001 - [ieeexplore.ieee.org](#)  
... However, they **may** not produce the minimal dissimilarity measures due to mismatching scales ... all 7 variants of every shape match each other better than they match **any** other shape ... each class contains an arbitrary number of objects, usually determined by **how many** models ...

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## Anomalies in the IBM acrith package

W Kahan, E LeBlanc - 1985 IEEE 7th Symposium on Computer ..., 1985 - [ieeexplore.ieee.org](#)  
... **How** reliable is ACRITH ... That should allay **any** qualms about accuracy ... The factor of 6 is roughly what S. Rump [5, pg.41] predicts from theoretical considerations, **Who** can use ACRITH? "...every FORTRAN programmer **who** is writing programs to solve linear and linearized ...

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## Stereo vision for planetary rovers: Stochastic modeling to near real-time implementation

L Matthies - International Journal of Computer Vision, 1992 - Springer  
... Estimates to subpixel resolution can be obtained in **many** ways ... We assume that **any** prior information about  $d$  comes from external sources, such as previous images, a laser ... It would be easy to compute posterior probabilities via (4) for both cases and to examine **how** peaked the ...

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## The inverse problem in microtectonics and the separation of tectonic phases

R Armijo, E Carey, A Cisternas - Tectonophysics, 1982 - Elsevier  
... The AC program was applied to the whole set of data without **any** knowledge of the ... The case of Milos Island shows **how** additional regional tectonic information **may** be used with the ... We would like to thank Paul Tapponnier for **manv** discussions and his constant encouragement ...

[\[PDF\] academia.edu](#)

# Sequence

Matrix Size	# of Matrices	# of Singular Matrices
$1 \times 1$	2	0
$2 \times 2$	16	8
$3 \times 3$	512	320
$4 \times 4$	65,536	43,264
$5 \times 5$	33,554,432	22,003,712
$6 \times 6$	68,719,476,736	43,090,149,376
$7 \times 7$	562,949,953,421,312	???

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# THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES<sup>®</sup>

founded in 1964 by N. J. A. Sloane

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(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

A057982	Number of singular $n \times n$ $(-1,1)$ -matrices.	1
	0, 8, 320, 43264, 22003712, 43090149376, 326720427917312, 9588057159626653696, 1086099857128493963804672 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> ; <a href="#">history</a> ; <a href="#">text</a> ; <a href="#">internal format</a> )	
OFFSET	1,2	
COMMENTS	$a(n) = 2^{(2n-1)*A046747(n-1)}$ . - Kevin Costello, May 18 2005	
LINKS	<a href="#">Table of <math>n, a(n)</math> for <math>n=1..9</math>.</a> R. P. Brent and J. H. Osborn, <a href="#">Bounds on minors of binary matrices</a> , arXiv preprint arXiv:1208.3330 [math.CO], 2012. - From <a href="#">N. J. A. Sloane</a> , Dec 25 2012 Konstantin Tikhomirov, <a href="#">Singularity of random Bernoulli matrices</a> , arXiv preprint arXiv:1812.09016 [math.PR], 2018-2019. Eric Weisstein's World of Mathematics, <a href="#">Singular Matrix</a> .	
FORMULA	$a(n)/2^{(n^2)} = (1/2 + o_n(1))^n$ (proved by Tikhomirov). - <a href="#">Timothy Y. Chow</a> , Jan 17 2019	
CROSSREFS	Complement of <a href="#">A056990</a> . Cf. <a href="#">A046747</a> . Sequence in context: <a href="#">A282621</a> <a href="#">A300189</a> <a href="#">A227657</a> * <a href="#">A041769</a> <a href="#">A209277</a> <a href="#">A182275</a>	

## How Many Distinct Eigenvalues?

For the Bohemian family of  $6 \times 6$  matrices with population  $\{-1, +1\}$ , how many **distinct eigenvalues** does the family contain?

# How Many Distinct Eigenvalues?

$$\begin{array}{l} > A, \text{Eigenvalues}(A) \\ \left[ \begin{array}{cccccc} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{array} \right], \left[ \begin{array}{l} 0 \\ \text{RootOf}(-Z^5 + 2Z^4 - 2Z^3 + 4Z^2 - 24Z + 32, \text{index} = 1) \\ \text{RootOf}(-Z^5 + 2Z^4 - 2Z^3 + 4Z^2 - 24Z + 32, \text{index} = 2) \\ \text{RootOf}(-Z^5 + 2Z^4 - 2Z^3 + 4Z^2 - 24Z + 32, \text{index} = 3) \\ \text{RootOf}(-Z^5 + 2Z^4 - 2Z^3 + 4Z^2 - 24Z + 32, \text{index} = 4) \\ \text{RootOf}(-Z^5 + 2Z^4 - 2Z^3 + 4Z^2 - 24Z + 32, \text{index} = 5) \end{array} \right] \end{array}$$

# How Many Distinct Eigenvalues?

► A. Eigenwert 0

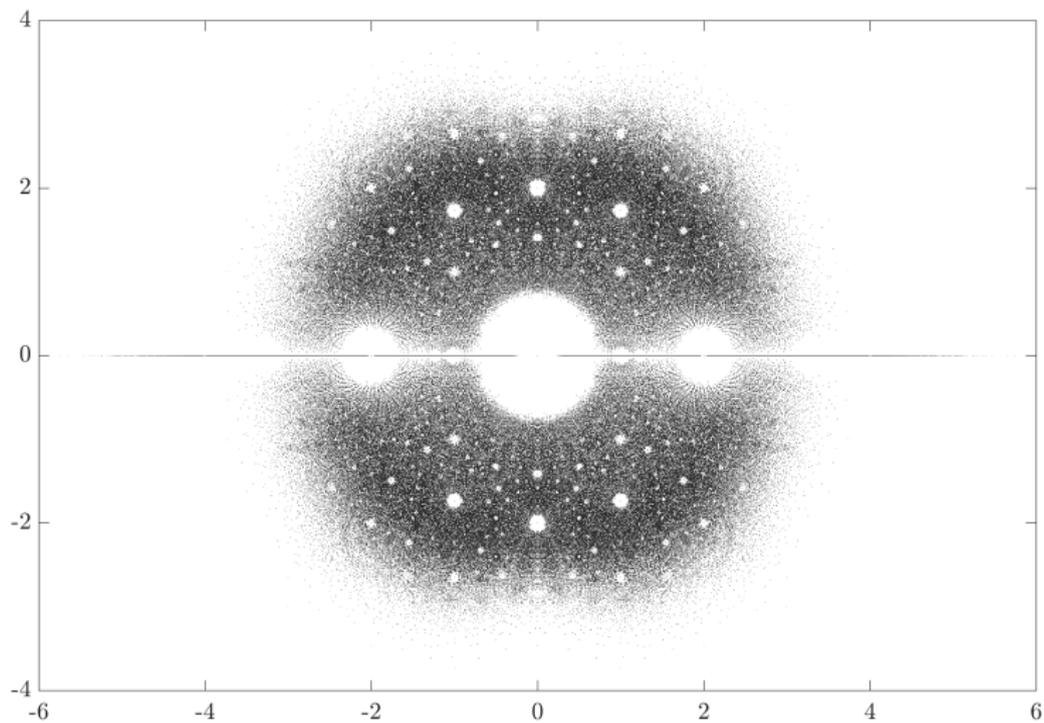
$$\frac{1}{2} \sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}} \pm \frac{\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}} + 38}}{(1+31\sqrt{982})^{11}} + 9\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}} + 19\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}}}$$

$$\frac{1}{2} \sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}} \pm \frac{\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}} + 38}}{(1+31\sqrt{982})^{11}} - 9\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}} + 19\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}}}$$

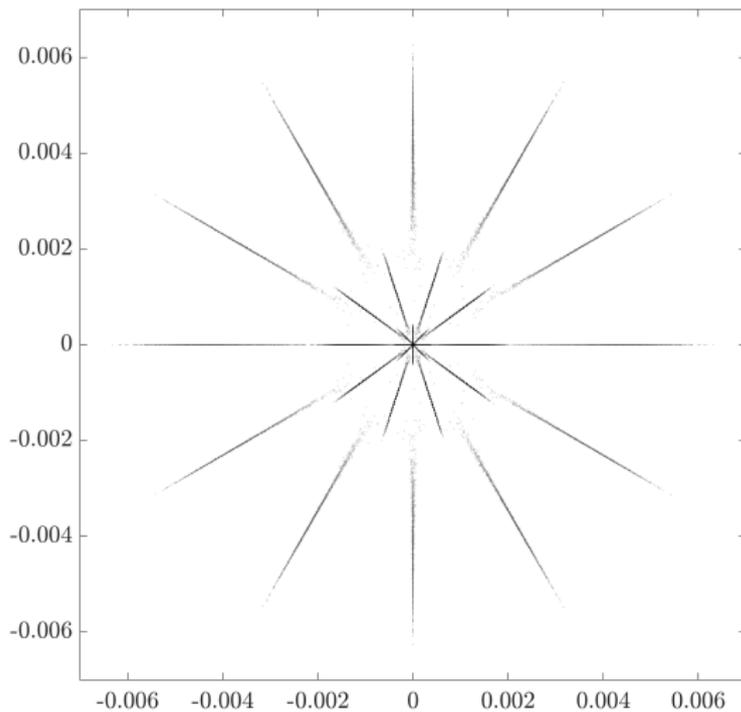
$$\frac{1}{2} \sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}} \pm \frac{\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}} + 38}}{(1+31\sqrt{982})^{11}} - 9\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}} - 19\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}}}$$

$$\frac{1}{2} \sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}} \pm \frac{\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}} + 38}}{(1+31\sqrt{982})^{11}} - 9\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}} - 19\sqrt{\frac{2(1+31\sqrt{982})^{11} - (1+31\sqrt{982})^{11} + 38}{(1+31\sqrt{982})^{11}}}}$$

# How Many Distinct Eigenvalues?



# How Many Distinct Eigenvalues?



# How Many Distinct Eigenvalues?

- Brute-force symbolic computation of eigenvalues is slow (20 years in Maple).
- Symbolic representations of eigenvalues can be difficult to work with.
- Numerical methods may not give correct answer due to numerical error in the eigenvalues.

# Counting Distinct Eigenvalues

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

# Counting Distinct Eigenvalues

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

Characteristic Polynomials:

$$p_1 = \det(xI - A_1) = x^6 + 2x^5 - 2x^4 + 4x^3 - 24x^2 + 32x$$

$$p_2 = \det(xI - A_2) = x^6 + 2x^5 + 2x^4 + 4x^3 + 8x^2$$

# Counting Distinct Eigenvalues

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

Characteristic Polynomials:

$$p_1 = \det(xI - A_1) = x^6 + 2x^5 - 2x^4 + 4x^3 - 24x^2 + 32x$$

$$p_2 = \det(xI - A_2) = x^6 + 2x^5 + 2x^4 + 4x^3 + 8x^2$$

Check if they **share** a root:

$$\gcd(p_1, p_2) = x$$

# Counting Distinct Eigenvalues

Factor into **irreducible** factors:

$$p_1 = x(x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32)$$

$$p_2 = x^2(x^4 + 2x^3 + 2x^2 + 4x + 8)$$

# Counting Distinct Eigenvalues

Factor into **irreducible** factors:

$$p_1 = x(x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32)$$

$$p_2 = x^2(x^4 + 2x^3 + 2x^2 + 4x + 8)$$

Make **square-free**:

$$q_1 = x(x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32)$$

$$q_2 = x(x^4 + 2x^3 + 2x^2 + 4x + 8)$$

# Counting Distinct Eigenvalues

Factor into **irreducible** factors:

$$p_1 = x(x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32)$$

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Make **square-free**:

$$q_1 = x(x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32)$$

$$q_2 = x(x^4 + 2x^3 + 2x^2 + 4x + 8)$$

Check if **coprime**:

$$\gcd(q_1, q_2) = x$$

# Counting Distinct Eigenvalues

Make coprime:

$$r_1 = \frac{q_1}{x} = x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32$$

$$r_2 = \frac{q_2}{x} = x^4 + 2x^3 + 2x^2 + 4x + 8$$

# Counting Distinct Eigenvalues

Make coprime:

$$r_1 = \frac{q_1}{x} = x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32$$
$$r_2 = \frac{q_2}{x} = x^4 + 2x^3 + 2x^2 + 4x + 8$$

The set  $\{\gcd(q_1, q_2), r_1, r_2\}$  defines a GCD-free basis of  $\{q_1, q_2\}$ .

$$\{x, x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32, x^4 + 2x^3 + 2x^2 + 4x + 8\}$$

# Counting Distinct Eigenvalues

**Input** : The set  $T$  of all characteristic polynomials of a Bohemian family.

$T_{\text{irred}} \leftarrow \emptyset.$

**for**  $t \in T$  **do**

    Let  $S$  be the set of irreducible factors of  $t$  over the integers.

$T_{\text{irred}} \leftarrow T_{\text{irred}} \cup S$

**end**

The number of distinct roots in  $T$  is  $\sum_{t \in T_{\text{irred}}} \deg(t).$

# Counting Distinct Eigenvalues

```
> T_irred := { } :  
> for t in T do  
    S := factors(t) :  
    S := map(p → p[1], S[2]) :  
    S := convert(S, set) :  
    T_irred := T_irred union S :  
end do :  
> T_irred := convert(T_irred, list) :  
> add(map(degree, T_irred));  
510743
```

# Counting Distinct Real Eigenvalues

```
> sturm_sequences := map(sturmseq, T_irred, x) :  
> real_root_count := map(sturm, sturm_sequences, x, -7, 7) :  
> add(real_root_count);  
145213
```

```
> real_roots := map(realroot, T_irred,  $\frac{1}{1000}$ ) :  
> add(map(nops, real_roots));  
145213
```

# Counting Distinct Eigenvalues

Matrix Size	# Char Polys	Eigenvalues	Real (Sturm)	Real (Descartes)
$1 \times 1$	2	<1ms	1ms	6ms
$2 \times 2$	6	<1ms	1ms	2ms
$3 \times 3$	28	5ms	1ms	5ms
$4 \times 4$	203	56ms	12ms	49ms
$5 \times 5$	3,150	1.372s	424ms	900ms
$6 \times 6$	131,641	87.2s	47.6s	84.4s

## Counting Distinct Eigenvalues

Matrix Size	# Matrices	# Eigenvalues	# Real Eigenvalues
$1 \times 1$	2	2	2
$2 \times 2$	16	9	5
$3 \times 3$	512	35	15
$4 \times 4$	65,536	335	119
$5 \times 5$	33,554,432	7,709	2,297
$6 \times 6$	68,719,476,736	510,743	145,213
$7 \times 7$	562,949,953,421,312	???	???

# The Characteristic Polynomial Database

The **Characteristic Polynomial Database** contains characteristic polynomials, minimal polynomials, and properties for a variety of Bohemian families.

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- 1,762,728,065 characteristic polynomials

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- 46 sequences on the OEIS (29 original)

The **Characteristic Polynomial Database** contains characteristic polynomials, minimal polynomials, and properties for a variety of Bohemian families.

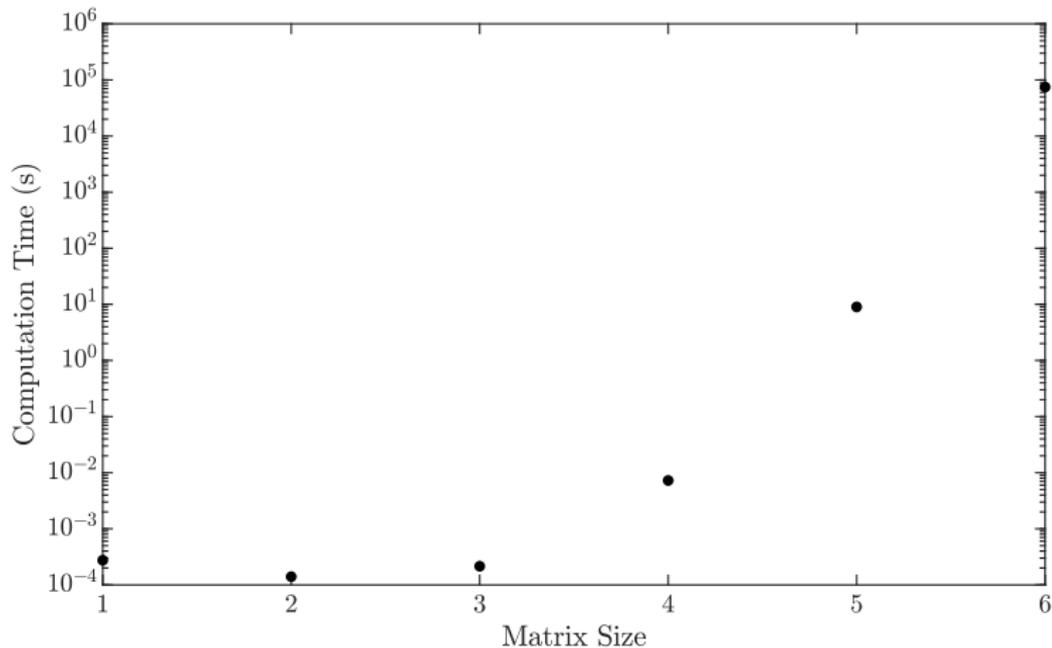
- 1,762,728,065 characteristic polynomials
- 2,366,960,967,336 matrices
- 46 sequences on the OEIS (29 original)
- 21 conjectures (6 proved, 1 disproved, 10 pending proofs)

# Computing Characteristic Polynomials

- C++ was used for computing characteristic polynomials and their frequencies
- Code generation in Maple was used for finding the coefficients of the characteristic polynomials
- Polynomials were stored as length  $n$  vectors of integers (`std::vector`)
- Polynomials were stored in a map (`std::map`)
  - The key was the vector of coefficients
  - The value was the number of occurrences of the polynomial

# Timing

Time to compute all characteristic polynomials for the family of  $n \times n$  matrices with population  $\{-1, +1\}$ .



Properties you **can** compute from characteristic polynomials:

- Number of singular/non-singular matrices
- Maximum determinant
- Number of unimodular matrices
- Number of nilpotent matrices
- Number of distinct characteristic polynomials
- Number of distinct eigenvalues
- Number of distinct real eigenvalues

Properties you **can't** compute from characteristic polynomials:

- Number of distinct minimal polynomials
- Number of non-derogatory matrices
- Number of distinct Jordan canonical forms
- Maximum/minimum condition number
- Number of normal matrices
- Number of rank  $k$  matrices for  $k < n$
- Number of positive definite matrices

# Counting Number of Distinct Jordan Canonical Forms

- Computing the Jordan Form requires a field extension
- Could use rational Jordan form<sup>2,3</sup>
- Count number of distinct Frobenius (rational) forms

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<sup>2</sup>Giesbrecht, “Nearly optimal algorithms for canonical matrix forms”.

<sup>3</sup>Kaltofen, Krishnamoorthy, and Saunders, “Fast parallel algorithms for similarity of matrices”.

## Number of Distinct Jordan Canonical Forms

Number of distinct Jordan canonical forms for the family of  $n \times n$  matrices with population  $\{-1, +1\}$ .

Matrix Size	# Matrices	Lower Bound	# Distinct JCFs	Upper Bound
$1 \times 1$	2	2	2	2
$2 \times 2$	16	6	6	7
$3 \times 3$	512	28	30	34
$4 \times 4$	65,536	203	232	266
$5 \times 5$	33,554,432	3,150	3,490	3,772
$6 \times 6$	68,719,476,736	131,641	???	144,138

**Conjecture 6:** The number of nilpotent  $n \times n$  matrices with entries from the set  $\{0,+1,+2\}$  is given by sequence A188457.

**Conjecture 8:** The maximum absolute determinant of an  $n \times n$  upper-Hessenberg matrix with entries from the set  $\{0,+1,+2\}$  and subdiagonal entries fixed at 1 is given by sequence A052542.

**Conjecture 11:** The maximum characteristic height of an  $n \times n$  upper-Hessenberg matrix with entries from the set  $\{0,+1,+2\}$ , subdiagonal entries fixed at 1, and diagonal entries fixed at 0 is given by sequence A058764.

- More families
  - Symmetric
  - Upper Triangular
  - Circulant
  - Tridiagonal
  - Toeplitz
- More properties
- Better algorithms

- `BohemianMatrices.com`
- `BohemianMatrices.com/cpdb`
- `twitter.com/bohemianmatrix`
- `github.com/BohemianMatrices`

Questions?