The Characteristic Polynomial Database

Steven E. Thornton

Western University

Definitions

A **family of Bohemian matrices** is a set of matrices of dimension n where the free entries are from the discrete set of integers of bounded height P called the **population** of the Bohemian family.

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Bohemian eigenvalues are the set of eigenvalues of a Bohemian family.

Example Family 1

The family of 6×6 matrices with population $\{-1, +1\}$.

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

Example Family 2

The family of 6×6 upper Hessenberg[1] matrices with population $\{-1, +1\}$ and subdiagonal entries fixed at 1.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

[1] Chan et al., “Bohemian Upper Hessenberg and Toeplitz Matrices”.

Questions

- How many matrices are singular?
- What is the maximum determinant?
- How many distinct characteristic polynomials does the family have?
- How many distinct eigenvalues does the family contain?
- How many distinct Jordan canonical are there?

How Many Matrices are Singular?

For the Bohemian family of 6×6 matrices with population $\{-1, +1\}$, how many matrices are singular?

How Many Matrices are Singular?

By brute-force computation on $2^{36} = 68{,}719{,}476{,}736$ matrices, there are 43,090,149,376 singular matrices.

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Maple	270 days

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Matlab	10 days
Julia	31 hours

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Julia	31 hours
Python (sequential)	20 days
Python (batched)	17 hours

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Matlab	10 days
Julia	31 hours
Python (sequential)	20 days
Python (batched)	17 hours
C++	4.75 hours

How Many Matrices are Singular?

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Language	Time
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Matlab	10 days
Julia	31 hours
Python (sequential)	20 days
Python (batched)	17 hours
C++	4.75 hours
CPDB	**124ms**

Symmetries

Symmetry	Description	Reduction
Transpose	If A is singular, A^T must also be singular.	2
Negation	If A is singular, $-A$ must also be singular.	2
Permutation	If A is singular, PAP^{-1} must also be singular for all permutation matrices P	$n!$

How Many Matrices are Singular?

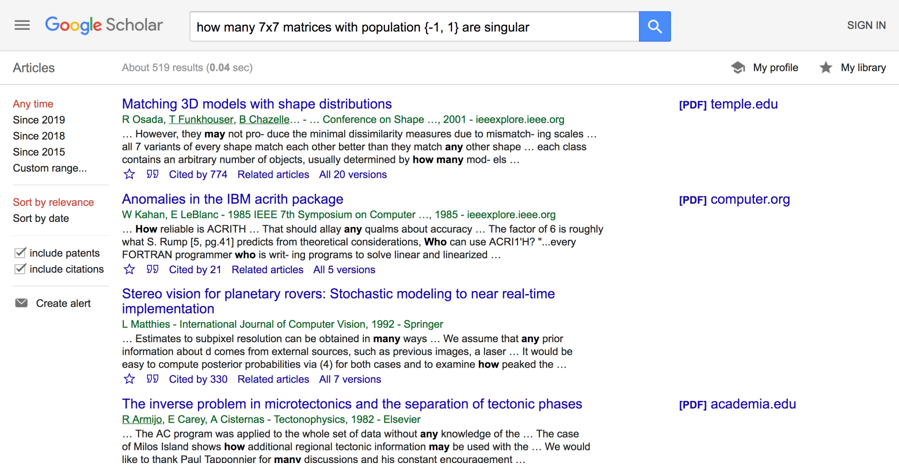
For the Bohemian family of 7×7 matrices with population $\{-1, +1\}$, how many matrices are singular?

How Many Matrices are Singular?

By brute-force computation on $2^{49} = 562{,}949{,}953{,}421{,}312$ matrices, there are ??? singular matrices.

Language	Time
Maple	7 millennia
Matlab	215 years
Julia	34 years
Python (sequential)	483 years
Python (batched)	21 years
C++	45 years

Literature Search?



Sequence

Matrix Size	# of Matrices	# of Singular Matrices
1×1	2	0
2×2	16	8
3×3	512	320
4×4	65,536	43,264
5×5	33,554,432	22,003,712
6×6	68,719,476,736	43,090,149,376
7×7	562,949,953,421,312	???

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A057982 Number of singular n X n (-1,1)-matrices. 1

0, 8, 320, 43264, 22003712, 43090149376, 326720427917312, 9588057159626653696, 1086099857128493963804672 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS a(n) = 2^(2n-1)*A046747(n-1). - Kevin Costello, May 18 2005

LINKS Table of n, a(n) for n=1..9.
R. P. Brent and J. H. Osborn, Bounds on minors of binary matrices, arXiv preprint arXiv:1208.3330 [math.CO], 2012. - From N. J. A. Sloane, Dec 25 2012
Konstantin Tikhomirov, Singularity of random Bernoulli matrices, arXiv preprint arXiv:1812.09016 [math.PR], 2018-2019.
Eric Weisstein's World of Mathematics, Singular Matrix.

FORMULA a(n)/2^(n^2) ~ (1/2 + o_n(1))^n (proved by Tikhomirov). - Timothy Y. Chow, Jan 17 2019

CROSSREFS Complement of A056990.
Cf. A046747.
Sequence in context: A282621 A300189 A227657 * A041769 A209277 A182275

How Many Distinct Eigenvalues?

For the Bohemian family of 6×6 matrices with population $\{-1, +1\}$, how many **distinct eigenvalues** does the family contain?

How Many Distinct Eigenvalues?

> $A, Eigenvalues(A)$

$$\begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ RootOf\left(\_Z^5 + 2\,\_Z^4 - 2\,\_Z^3 + 4\,\_Z^2 - 24\,\_Z + 32, index = 1\right) \\ RootOf\left(\_Z^5 + 2\,\_Z^4 - 2\,\_Z^3 + 4\,\_Z^2 - 24\,\_Z + 32, index = 2\right) \\ RootOf\left(\_Z^5 + 2\,\_Z^4 - 2\,\_Z^3 + 4\,\_Z^2 - 24\,\_Z + 32, index = 3\right) \\ RootOf\left(\_Z^5 + 2\,\_Z^4 - 2\,\_Z^3 + 4\,\_Z^2 - 24\,\_Z + 32, index = 4\right) \\ RootOf\left(\_Z^5 + 2\,\_Z^4 - 2\,\_Z^3 + 4\,\_Z^2 - 24\,\_Z + 32, index = 5\right) \end{bmatrix}$$

How Many Distinct Eigenvalues?

> *A, Eigenvalues*(*A*)

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
-\frac{1}{2} + \frac{\sqrt{3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}}{6} + \frac{\sqrt{-\frac{6\left(\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}\,(1+3I\sqrt{762})^{2/3} + 9\sqrt{3}\,(1+3I\sqrt{762})^{1/3} + (1+3I\sqrt{762})^{1/3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}} + 19\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}\right)}{(1+3I\sqrt{762})^{1/3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}}}}{6} \\
-\frac{1}{2} + \frac{\sqrt{3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}}{6} - \frac{\sqrt{-\frac{6\left(\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}\,(1+3I\sqrt{762})^{2/3} + 9\sqrt{3}\,(1+3I\sqrt{762})^{1/3} + (1+3I\sqrt{762})^{1/3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}} + 19\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}\right)}{(1+3I\sqrt{762})^{1/3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}}}}{6} \\
-\frac{1}{2} - \frac{\sqrt{3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}}{6} + \frac{\sqrt{6}\sqrt{\frac{-\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}\,(1+3I\sqrt{762})^{2/3} + 9\sqrt{3}\,(1+3I\sqrt{762})^{1/3} - (1+3I\sqrt{762})^{1/3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}} - 19\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}}{(1+3I\sqrt{762})^{1/3}\sqrt{\frac{2(1+3I\sqrt{762})^{2/3} - (1+3I\sqrt{762})^{1/3} + 38}{(1+3I\sqrt{762})^{1/3}}}}}}{6} \\
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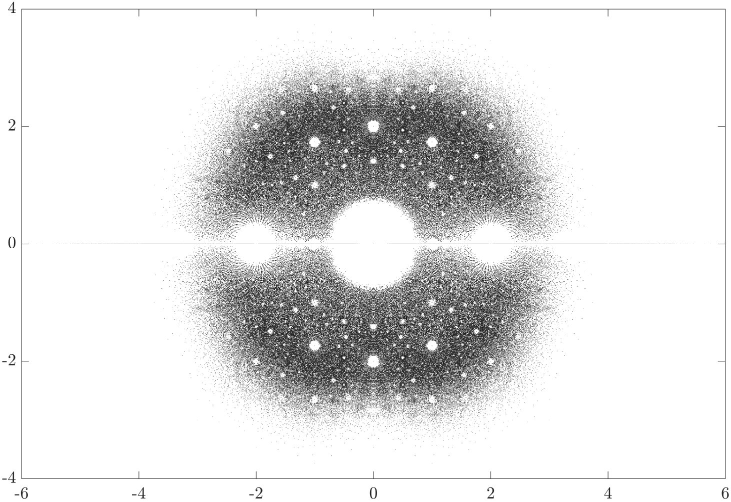
\end{bmatrix}
$$

How Many Distinct Eigenvalues?



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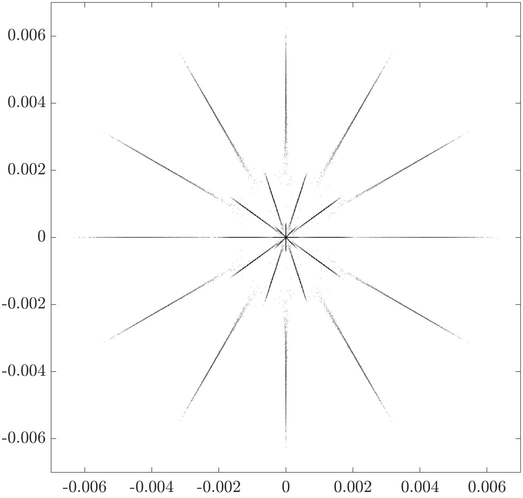

How Many Distinct Eigenvalues?

- Brute-force symbolic computation of eigenvalues is slow (20 years in Maple).
- Symbolic representations of eigenvalues can be difficult to work with.
- Numerical methods may not give correct answer due to numerical error in the eigenvalues.

Counting Distinct Eigenvalues

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

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Characteristic Polynomials:

$$p_1 = \det(xI - A_1) = x^6 + 2x^5 - 2x^4 + 4x^3 - 24x^2 + 32x$$
$$p_2 = \det(xI - A_2) = x^6 + 2x^5 + 2x^4 + 4x^3 + 8x^2$$

Counting Distinct Eigenvalues

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

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Check if they **share** a root:

$$\gcd(p_1, p_2) = x$$

Counting Distinct Eigenvalues

Factor into **irreducible** factors:

$$p_1 = x(x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32)$$
$$p_2 = x^2(x^4 + 2x^3 + 2x^2 + 4x + 8)$$

Counting Distinct Eigenvalues

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Make **square-free**:

$$q_1 = x(x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32)$$
$$q_2 = x(x^4 + 2x^3 + 2x^2 + 4x + 8)$$

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Check if **coprime**:

$$\gcd(q_1, q_2) = x$$

Counting Distinct Eigenvalues

Make coprime:

$$r_1 = \frac{q_1}{x} = x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32$$
$$r_2 = \frac{q_2}{x} = x^4 + 2x^3 + 2x^2 + 4x + 8$$

Counting Distinct Eigenvalues

Make coprime:

$$r_1 = \frac{q_1}{x} = x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32$$
$$r_2 = \frac{q_2}{x} = x^4 + 2x^3 + 2x^2 + 4x + 8$$

The set $\{\text{gcd}(q_1, q_2), r_1, r_2\}$ defines a GCD-free basis of $\{q_1, q_2\}$.

$$\{x, x^5 + 2x^4 - 2x^3 + 4x^2 - 24x + 32, x^4 + 2x^3 + 2x^2 + 4x + 8\}$$

Counting Distinct Eigenvalues

Input : The set T of all characteristic polynomials of a Bohemian family.

$T_{\mathrm{irred}} \leftarrow \varnothing$.
for $t \in T$ **do**
  Let S be the set of irreducible factors of t over the integers.
  $T_{\mathrm{irred}} \leftarrow T_{\mathrm{irred}} \cup S$
end

The number of distinct roots in T is $\sum_{t \in T_{\mathrm{irred}}} \deg(t)$.

Counting Distinct Eigenvalues

```
> T_irred := { } :
> for t in T do
      S := factors(t) :
      S := map(p→p[1], S[2]) :
      S := convert(S, set) :
     T_irred := T_irred union S :
  end do:
> T_irred := convert(T_irred, list) :
> add(map(degree, T_irred));
                              510743
```

Counting Distinct Real Eigenvalues

```
> sturm_sequences := map(sturmseq, T_irred, x) :
> real_root_count := map(sturm, sturm_sequences, x, -7, 7) :
> add(real_root_count);
                          145213
```

```
> real_roots := map(realroot, T_irred, 1/1000) :
> add(map(nops, real_roots));
                          145213
```

Counting Distinct Eigenvalues

Matrix Size	# Char Polys	Eigenvalues	Real (Sturm)	Real (Descartes)
1×1	2	<1ms	1ms	6ms
2×2	6	<1ms	1ms	2ms
3×3	28	5ms	1ms	5ms
4×4	203	56ms	12ms	49ms
5×5	3,150	1.372s	424ms	900ms
6×6	131,641	87.2s	47.6s	84.4s

Counting Distinct Eigenvalues

Matrix Size	# Matrices	# Eigenvalues	# Real Eigenvalues
1×1	2	2	2
2×2	16	9	5
3×3	512	35	15
4×4	65,536	335	119
5×5	33,554,432	7,709	2,297
6×6	68,719,476,736	510,743	145,213
7×7	562,949,953,421,312	???	???

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- 46 sequences on the OEIS (29 original)

The Characteristic Polynomial Database

The **Characteristic Polynomial Database** contains characteristic polynomials, minimal polynomials, and properties for a variety of Bohemian families.

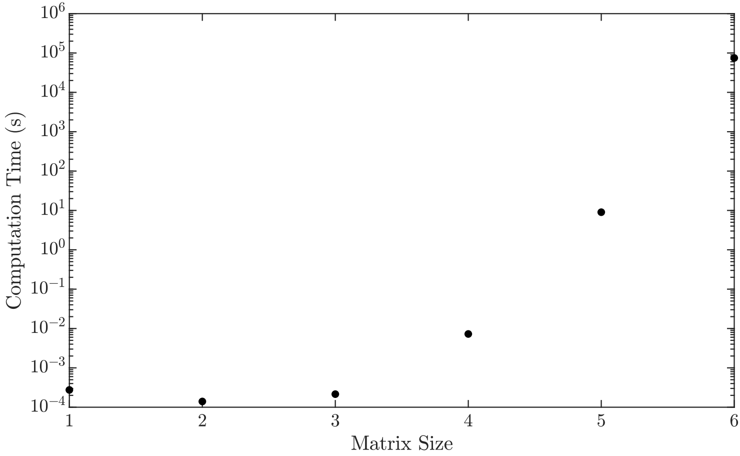
- 1,762,728,065 characteristic polynomials
- 2,366,960,967,336 matrices
- 46 sequences on the OEIS (29 original)
- 21 conjectures (6 proved, 1 disproved, 10 pending proofs)

Computing Characteristic Polynomials

- C++ was used for computing characteristic polynomials and their frequencies
- Code generation in Maple was used for finding the coefficients of the characteristic polynomials
- Polynomials were stored as length *n* vectors of integers (`std::vector`)
- Polynomials were stored in a map (`std::map`)
  - The key was the vector of coefficients
  - The value was the number of occurrences of the polynomial

Timing

Time to compute all characteristic polynomials for the family of $n \times n$ matrices with population $\{-1, +1\}$.



Properties

Properties you **can** compute from characteristic polynomials:

- Number of singular/non-singular matrices
- Maximum determinant
- Number of unimodular matrices
- Number of nilpotent matrices
- Number of distinct characteristic polynomials
- Number of distinct eigenvalues
- Number of distinct real eigenvalues

Properties

Properties you **can't** compute from characteristic polynomials:

- Number of distinct minimal polynomials
- Number of non-derogatory matrices
- Number of distinct Jordan canonical forms
- Maximum/minimum condition number
- Number of normal matrices
- Number of rank k matrices for $k < n$
- Number of positive definite matrices

Counting Number of Distinct Jordan Canonical Forms

- Computing the Jordan Form requires a field extension
- Could use rational Jordan form[2,3]
- Count number of distinct Frobenius (rational) forms

[2] Giesbrecht, “Nearly optimal algorithms for canonical matrix forms”.
[3] Kaltofen, Krishnamoorthy, and Saunders, “Fast parallel algorithms for similarity of matrices”.

Number of Distinct Jordan Canonical Forms

Number of distinct Jordan canonical forms for the family of $n \times n$ matrices with population $\{-1, +1\}$.

Matrix Size	# Matrices	Lower Bound	# Distinct JCFs	Upper Bound
1×1	2	2	2	2
2×2	16	6	6	7
3×3	512	28	30	34
4×4	65,536	203	232	266
5×5	33,554,432	3,150	3,490	3,772
6×6	68,719,476,736	131,641	???	144,138

Conjectures

Conjecture 6: The number of nilpotent $n \times n$ matrices with entries from the set $\{0,+1,+2\}$ is given by sequence A188457.

Conjecture 8: The maximum absolute determinant of an $n \times n$ upper-Hessenberg matrix with entries from the set $\{0,+1,+2\}$ and subdiagonal entries fixed at 1 is given by sequence A052542.

Conjecture 11: The maximum characteristic height of an $n \times n$ upper-Hessenberg matrix with entries from the set $\{0,+1,+2\}$, subdiagonal entries fixed at 1, and diagonal entries fixed at 0 is given by sequence A058764.

Future Work

- More families
  - Symmetric
  - Upper Triangular
  - Circulant
  - Tridiagonal
  - Toeplitz
- More properties
- Better algorithms

Links

- BohemianMatrices.com
- BohemianMatrices.com/cpdb
- twitter.com/bohemianmatrix
- github.com/BohemianMatrices

Questions?